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SINGULARITIES AT THE CONTACT POINT OF TWO KISSING NEUMANN BALLS

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Abstract. We investigate eigenfunctions of the Neumann Laplacian in a bounded domain $\Omega \subset \mathbb{R}^d$, where a cuspidal singularity is caused by a cavity consisting of two touching balls, or discs in the planar case. We prove that the eigenfunctions with all of their derivatives are bounded in $\overline{\Omega}$, if the dimension d equals 2, but in dimension $d \geq 3$ their gradients have a strong singularity $O(|x - \mathcal{O}|^{-\alpha})$, $\alpha \in (0, 2 - \sqrt{2}]$ at the point of tangency $\mathcal{O}$. Our study is based on dimension reduction and other asymptotic procedures, as well as the Kondratiev theory applied to the limit differential equation in the punctured hyperplane $\mathbb{R}^{d-1} \setminus \mathcal{O}$. We also discuss other shapes producing thinning gaps between touching cavities.

1. Introduction.

1.1. **Prelude.** Eigenfunctions of the Dirichlet and Neumann problems for the Laplace operator in a domain $\Omega \subset \mathbb{R}^d$ with smooth boundary $\partial\Omega$ are infinitely differentiable in the closure $\overline{\Omega} = \Omega \cup \partial\Omega$. However, if the boundary is irregular, for example, it has a corner or conical point, an eigenfunction may, and usually does, behave "badly" so that it only belongs to the Sobolev space $H^1(\Omega)$ instead of $H^2(\Omega)$. Singularities of solutions of elliptic boundary value problems in domains having irregular submanifolds on the boundary play an important role in applications. We recall the theory of brittle fractures [5] with square-root singularities of stresses in a plate at the crack tip; high voltage electrostatics with Wiener criterion [26] on continuity of harmonics, which is mythically linked with Saint Elmo's fires; black holes for vibrations at cuspidal irregularities of the boundary, which are caused by oscillatory behaviour of the solutions [19] and which enable wave process in tapering elastic bodies, a phenomenon with engineering applications as dampers in acoustic and elastic waves.

The Kondratiev theory [9] of weighted Sobolev spaces, see also [6, 22, 10] and others, provides the tools for studying elliptic boundary problems in domains with corner and conical points, edges and polyhedral submanifolds. Moreover, a domain with a cuspidal peak can be transformed using an appropriate change of variables into a conical and eventually a cylindrical shape so that the Kondratiev theory applies again. However, the resulting differential operators are "strongly" perturbed and their treatment requires improved techniques [17], see also [11, 2] for asymptotic formulas for solutions at cuspidal peak tops in the linearized elasticity system and

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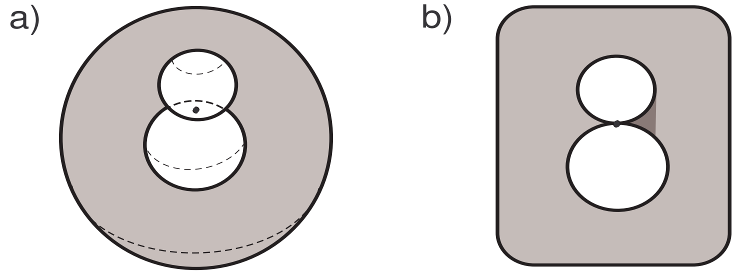

FIGURE 1.1. Exterior domain of two kissing balls a) in $d = 3$, b) in $d = 2$.

[24, 7, 8, 14] for the linear water-wave equation in domains with cuspidal points and edges.

There are several types of naturally occurring boundary irregularities which have not yet been studied and which require novel modifications to the multi-step procedures including Kondratiev's techniques. One example of such an isolated singularity consists of the point of tangency $\mathcal{O}$ of two "kissing" balls in the exterior domain of the balls, see Fig. 1.1. In this paper we investigate the spectral Neumann problem for the Laplace operator and prove that in dimension $d \geq 3$ the gradients of the eigenfunctions have a rather strong singularity $O(|x|^{-\alpha})$, $\alpha \in (0, 2-\sqrt{2}]$ at the point $\mathcal{O}$ (the most singular case being $\alpha = 2 - \sqrt{2} \approx 0.586$ for $d = 3$). On the contrary, in dimension $d = 2$, where the discs in Fig. 1.1.b) form two cusps with a common top $\mathcal{O}$, any eigenfunction and all its derivatives are bounded in the domain but may have a discontinuity at $\mathcal{O}$. These statements are direct consequences of our main result, Theorem 3.7, which gives an asymptotic representation of the eigenfunctions.

In the analogous Dirichlet problem all eigenfunctions decay exponentially as $x \to \mathcal{O}$ and hence they are infinitely smooth in the closure of the domain, cf. Section 1.3. In the geometric situation[1] of Fig. 1.2, a) and b), the Steklov spectral problem describes the propagation of surface waves over a heavy ideal liquid, cf. [12], and it has oscillatory solutions in both natural dimensions $d = 3$ and $d = 2$, see [24] and [7], respectively. We thus see that the Neumann problem is exceptional because its eigenfunctions have absolutely different behaviour, as they are smooth in separated closed cusps in $d = 2$, but for $d \geq 3$ they get much stronger singularities than even those in the mechanics of cracks, see Sections 2.5 and 2.2, respectively.

We will briefly discuss some generalizations of our results in Section 4. Our scheme for the examination of singularities works in much more general situations and some particular cases will be mentioned in the final section. However, in the case the principal curvatures are the same for the two cavities at the touching point $\mathcal{O}$, releasing the requirement of exact rotational symmetry would lead to the situation that the main singularity $O(|x|^{\alpha})$ would stay the same, but the logarithmic terms of the eigenfunctions might not disappear in dimension $d \geq 3$ and the infinite differentiability in dimension $d = 2$ might be lost. The symmetry is also broken in the case of touching ellipsoids, see Fig. 1.3 a). Here, the method of the present work would lead to a limit differential equation which has variable coefficients in the spherical coordinates θ so that the simple trick in Section 2.2 would not apply, nor could

[1]For the Navier-Stokes equation, the singularity of the velocity field and pressure in a viscous fluid around a ball on the planar bottom were examined in [21]

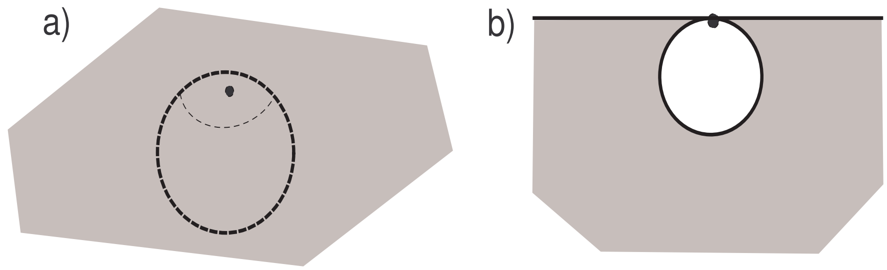

FIGURE 1.2. Geometry for the corresponding Steklov problem a) in $d = 3$, b) in $d = 2$.

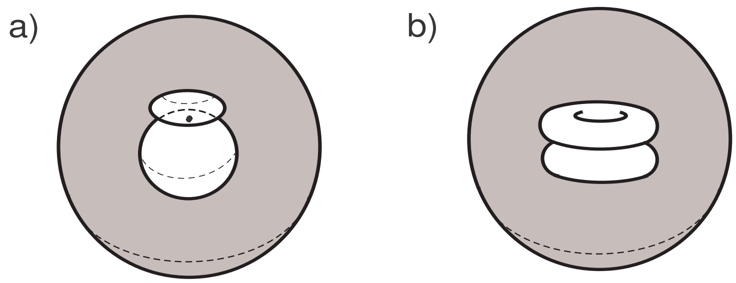

FIGURE 1.3. a) Tangential ellipsoids, b) tangential tori.

the explicit solutions be presented. Thus, for ellipsoidal cavities the precise form of the main singularity of the eigenfunctions remains unsolved. Finally, an example in Section 4.3 shows how the singularity of the eigenfunction can be strengthened further by perfecting the tangency.

1.2. **Problem setting.** Two balls

$$B^{\pm} = \{x = (y, z) \in \mathbb{R}^{d-1} \times \mathbb{R} \,:\, |y|^2 + |z \mp R_{\pm}|^2 < R_{\pm}^2\} \tag{1.1}$$

touch each other at the origin $\mathcal{O} = \{0\}$ of the Cartesian coordinate system $x = (x_1, \ldots, x_d)$ of the Euclidean space $\mathbb{R}^d$, $d \geq 2$. We assume that $R_- \geq R_+ > 0$ and set $R = R_+$. Let $\Omega^\circ \subset \mathbb{R}^d$ be a bounded domain which contains both balls (1.1) and has a smooth (for simplicity, C^∞) boundary Γ°. We introduce the domain in Fig. 1.1,

$$\Omega = \Omega^\circ \setminus \left(\overline{B^+} \cup \overline{B^-}\right) \tag{1.2}$$

with the boundary $\partial\Omega = \Gamma^\circ \cup \Gamma^+ \cup \Gamma^-$, where $\Gamma^{\pm} = \partial B^{\pm}$ are spheres and $\mathcal{O} = \Gamma^+ \cap \Gamma^-$.

We consider the Neumann problem in the domain (1.2),

$$-\Delta_x u(x) = \lambda u(x) \;, \quad x \in \Omega, \tag{1.3}$$

$$\partial_\nu u(x) = 0 \;, \quad x \in \Gamma \setminus \mathcal{O} \;, \tag{1.4}$$

where $\Delta_x = \nabla_x \cdot \nabla_x$, ∇_x is the gradient, the central dot stands for the scalar product, ∂_ν is the outward normal derivative and λ is the spectral parameter. The variational formulation of this problem reads as

$$(\nabla_x u, \nabla_x v)_\Omega = \lambda (u, v)_\Omega \quad \forall\, v \in H^1(\Omega). \tag{1.5}$$

Here, $(\ ,\ )_\Omega$ is the natural inner product in the Lebesgue space $L^2(\Omega)$ and $H^1(\Omega)$ is the Sobolev space. According to [15, Sec. 1.4.6], the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, and the spectrum of the problem (1.5) is discrete and consists of the eigenvalue sequence

$$0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \ldots \le \lambda_n \le \ldots \to +\infty.$$

The corresponding eigenfunctions $u_n \in H^1(\Omega)$ are infinitely differentiable in $\overline{\Omega} \setminus \mathcal{O}$ and therefore satisfy the differential problem (1.3)–(1.4).

In Section 4 we will discuss possible generalizations, in particular the case $R_- < 0$, Fig. 4.1.a) and b), which corresponds to nested kissing balls. Our analysis applies of course to the geometry in Fig. 1.2.a) and b), when formally $R_+ = -\infty$ but $R_- > 0$ is finite.

1.3. **Remarks on the Dirichlet problem.** Let us replace for a while the Neumann condition (1.4) by

$$u(x) = 0 \quad , \quad x \in \Gamma \setminus \mathcal{O}. \tag{1.6}$$

The variational formulation of the problem (1.3), (1.6) takes the form (1.5) but the Sobolev space $H^1(\Omega)$ is replaced by its subspace

$$H^1_0(\Omega) = \{v \in H^1(\Omega) \,:\, v = 0 \text{ on } \partial\Omega\}.$$

Since the thickness $H(y) = H_+(y) + H_-(y)$ of the "gap"

$$\Pi = \big\{x \,:\, r = |y| < R\,, z \in \big(-H_-(y), H_+(y)\big)\big\} \tag{1.7}$$

between the balls B_+ and B_- decays at the rate $O(r^2)$ as $x \to \mathcal{O}$, the Dirichlet condition yields the weighted Friedrichs inequality

$$\int_\Pi H(y)^{-2}|u(x)|^2 dx \le \frac{1}{\pi^2}\int_\Pi \Big|\frac{\partial u}{\partial z}(x)\Big|^2 dx. \tag{1.8}$$

One can use (1.8) and the same argument as in [18, 23, 4] to derive the following estimate for a weighted norm of eigenfunctions; we give a sketch of the proof and further references for the convenience of the reader. The eigenfunctions u_n are denoted in the same way as for the Neumann problem.

Proposition 1.1. *There exists $\beta > 0$ such that the weighted norm $\big\||x|^{-2}e^{\beta/|x|}u_n; L^2(\Omega)\big\|$ is finite for all eigenfunctions $u_n \in H^1_0(\Omega)$ of the problem* (1.3), (1.6).

Proof. We insert into the integral identity (1.5) (corresponding to the problem (1.3), (1.6)) the test function $v = U_n \in H^1_0(\Omega)$, where $U_n = R_\varrho u_n$ and the weight function R_ϱ is equal to $e^{\beta/R}$ in $\Omega \setminus \Pi$ but

$$R_\varrho(x) = \begin{cases} e^{\beta/r} \text{ for } R > |y| > \varrho \\ e^{\beta/\varrho} \text{ for } |y| < \varrho \end{cases} \qquad \text{in } \Pi. \tag{1.9}$$

The parameter $\varrho > 0$ is small and will be sent to 0. Notice that $\nabla_x R_\varrho = 0$ in $\Omega \setminus \Pi$ and $|\nabla_x R_\varrho(x)| \le \beta r^{-2} R_\varrho(x)$ for $x \in \Pi$. Simple transformations and inequality (1.8) show that

$$\begin{aligned} \lambda\|U_n; L^2(\Omega)\|^2 &= \|\nabla_x U_n; L^2(\Omega)\|^2 - \|U_n R_\varrho^{-1}\nabla_x R_\varrho; L^2(\Pi)\|^2 \\ &\ge \pi^{-2}\|H^{-1}U_n; L^2(\Pi)\|^2 - \beta^2\|r^{-2}U_n; L^2(\Pi)\|^2. \end{aligned} \tag{1.10}$$

According to (1.1) we have

$$H_\pm(y) = R_\pm^2 - \sqrt{R_\pm^2 - |y|^2} = \frac{|y|^2}{2R_\pm} + O(|y|^4). \tag{1.11}$$

Therefore we can fix $\beta > 0$ such that the norm $\|r^{-2}R_\varrho u_n; L^2(\Pi)\|$ becomes uniformly bounded in $\varrho \in (0, R)$. Moreover, the weight (1.9) is monotone increasing as $\varrho \to +0$, and this limit passage completes the proof. $\boxtimes$

In Section 3.4 we will explain how to convert the estimate of the weighted L^2-norm in Proposition 1.1 into an estimate of an exponentially weighted Hölder norm and conclude that $u_n \in C^\infty(\overline{\Omega})$ in the Dirichlet problem (1.3), (1.6).

Remark 1.2. If the domain Ω is symmetric with respect to the hyperplane $\{x : z = 0\}$ and in particular $R_+ = R_-$, then some eigenfunctions of the problem (1.3), (1.4) are odd in z and hence the inequality (1.8) and Proposition 1.1 are still valid for them. Thus, the smoothness of certain Neumann eigenfunctions can be verified in a simple way.

We finally mention the paper [3], where it was observed that the solution of the Dirichlet problem for the Poisson equation $-\Delta u = f$ in a two-dimensional cuspidal domain inherits the infinite differentiability of the right-hand side, if f is infinitely differentiable in $\overline{\Omega}$.

1.4. **Structure of the paper.** In Section 2 we will present the dimension reduction and related asymptotic procedures as well as examine the solutions of the limit degenerate differential equation in $\mathbb{R}^{d-1}_\bullet = \mathbb{R}^{d-1} \setminus \mathcal{O}$, especially in dimension $d = 2$. Section 3 is devoted to the justification of the constructed formal asymptotic expansions, in particular by applying the basic tools of Kondratiev theory [9]. The results, the weighted Sobolev and Hölder norm estimates of the asymptotic remainders, allow us to conclude the smoothness properties of the eigenfunctions of (1.3), (1.4) and to distinguish between the cases $d \geq 3$ and $d = 2$. Finally, in Section 4 we briefly discuss some variations of the shapes, including contacts of ellipsoids, paraboloids and torii, cf. Fig. 1.3 and 4.1.

2. Formal asymptotic analysis.

In this section we perform preparatory work and propose formal asymptotic series which leave in the equation (1.3) and in the boundary condition (1.4) discrepancies decaying at any given order $O(|x|^\alpha)$ as $x \to \mathcal{O}$. The main conclusions of our analysis are formulated in Section 2.2 and 2.5 in dimensions $d \geq 3$ and $d = 2$, respectively. The desired theorems on asymptotics will be derived in the next section, after justifying the asymptotic forms obtained in Section 2.

2.1. **Asymptotic ansätze.** Since the thickness of the gap Π, (1.7), decreases when x approaches $\mathcal{O}$ and vanishes at the limit, see (1.11) and Fig. 1.1, it can be considered as a thin domain near the coordinate origin and we can apply a standard asymptotic procedure, cf. [16, Ch. 13], which is rather simple as the equation is scalar valued. However, compared with the traditional application of dimension reduction in the theory of plates, there is a crucial difference, namely, the limit problem is posed in the punctured hyperplane $\mathbb{R}^{d-1} \setminus \mathcal{O}$ and it is not uniquely solvable in any sense. Hence, the formal asymptotics will involve unknown coefficients, the determination of which will be postponed to Section 3 and will be based on a completely different argument.

To perform the formal dimension reduction in the problem (1.3), (1.4), we assume that the main asymptotic term of the mean value function

$$\mathbf{u}(y) = \frac{1}{H(y)} \int_{-H_-(y)}^{H_+(y)} u(y,z)dz \quad , \quad y \in \mathbf{B} := \{y \in \mathbb{R}^{d-1} : \; r = |y| < R\}, \tag{2.1}$$

is a power-law solution centred at the coordinate origin

$$U^0(y) = r^\Lambda \Phi^0(\theta). \tag{2.2}$$

Here, $(r,\theta) \in \mathbb{R}_+ \times \mathbb{S}^{d-2}$ are the spherical coordinates, while the number Λ and the function Φ^0 on the unit sphere $\mathbb{S}^{d-2} \subset \mathbb{R}^{d-1}$ are to be determined. The difference

$$u_\perp(y,z) = u(y,z) - \mathbf{u}(y) \tag{2.3}$$

satisfies the orthogonality condition

$$\int_{-H_-(y)}^{H_+(y)} u_\perp(y,z)dz = 0 \ , \quad y \in \mathbf{B}, \tag{2.4}$$

and therefore the Poincaré inequality

$$\frac{1}{H(y)^2} \int_{-H_-(y)}^{H_+(y)} \big| u_\perp(y,z)\big|^2 dz \le \frac{1}{\pi^2} \int_{-H_-(y)}^{H_+(y)} \big|\partial_z u_\perp(y,z)\big|^2 dz = \frac{1}{\pi^2} \int_{-H_-(y)}^{H_+(y)} \big|\partial_z u(y,z)\big|^2 dz. \tag{2.5}$$

The left integral in (2.5) has the factor $H(y)^{-2} = O(r^{-4})$ appearing also in (1.8), and this suggests the form

$$U^1(y,\zeta) = r^{\Lambda+2}\Phi^1(\theta,\zeta) \tag{2.6}$$

for the correction term in the asymptotic ansatz for an eigenfunction,

$$u(x) = U^0(y) + U^1(y,\zeta) + \dots \, . \tag{2.7}$$

Here, the function Φ^1 of the variables $(\theta,\zeta) \in \mathbb{S}^{d-2} \times \Upsilon$ is to be found, and ζ is a stretched coordinate,

$$\zeta = H(y)^{-1}(z - h(y)) \in \Upsilon = (-1/2, 1/2) \ , \tag{2.8}$$

$$h(y) = \frac{1}{2}\big(H_+(y) - H_-(y)\big) \ . \tag{2.9}$$

We insert the asymptotic ansatz (2.7) into the restriction of (1.3) to Π, take into account that

$$\begin{aligned} &\nabla_y\big[U^1\big(y, H(y)^{-1}(z-h(y))\big)\big] \\ &= \nabla_y U^1(y,\zeta) - H(y)^{-1}\big(\zeta\nabla_y H(y) + \nabla_y h(y)\big)\partial_\zeta U^1(y,\zeta) \ , \\ &\partial_z U^1\big(y, H(y)^{-1}(z-h(y))\big) = H(y)^{-1}\partial_\zeta U^1(y,\zeta) \ , \end{aligned} \tag{2.10}$$

and collect terms of order $r^{\Lambda-2}$. Owing to (1.11), (2.10) and (2.2), (2.6), we obtain the equation

$$-\partial_\zeta^2 U^1(y,\zeta) = H^0(y)^2 \Delta_y U^0(y) \ , \quad \zeta \in \Upsilon, \tag{2.11}$$

where we put

$$H^0(y) = \frac{1}{2}\big(R_+^{-1} + R_-^{-1}\big)|y|^2 =: A^0 r^2 \ ,$$

$$h^0(y) = \frac{1}{4}\big(R_+^{-1} - R_-^{-1}\big)|y|^2 =: a^0 r^2, \tag{2.12}$$

and changed the thickness function $H(y)$ to its principal term $H^0(y)$ according to (1.11). The normal derivatives $\partial_{\nu^\pm}$ on the lateral sides $\varpi_\pm = \{x : y \in \mathbf{B},\ z = \pm H_\pm(y)\}$ have the form

$$\partial_{\nu^\pm} = J_\pm(y)^{-1}\big(\pm\partial_z - \nabla_y H_\pm(y)\cdot\nabla_y\big) \ , \quad J_\pm(y) = \big(1 + |\nabla_y H_\pm(y)|^2\big)^{1/2}. \tag{2.13}$$

Hence, in view of (2.13), (2.2), (2.6), the boundary condition (1.4) restricted to $\varpi_\pm$ turns after the replacement $H_\pm(y) \mapsto H_\pm^0(y) = A_\pm^0|y|^2$ into

$$\pm\partial_\zeta U^1(y, \pm 1/2) = H^0(y)\nabla_y H_\pm^0(y)\cdot\nabla_y U^0(y). \tag{2.14}$$

In the Neumann problem (2.11), (2.14), $y \in \mathbf{B}$ is considered as a parameter, and its compatibility condition

$$\int_\Upsilon H^0(y)^2\Delta_y U^0(y)d\zeta + H^0(y)\sum_\pm \nabla_y H_\pm^0(y)\cdot\nabla_y U^0(y) = 0 \tag{2.15}$$

is converted into the limit differential equation

$$-\nabla_y\cdot\big(H^0(y)\nabla_y U^0(y)\big) = 0 \ , \quad y \in \mathbb{R}_\bullet^{d-1} = \mathbb{R}^{d-1}\setminus\mathcal{O}. \tag{2.16}$$

Finally, we write

$$\begin{aligned} U_\perp^1(y,\zeta) = r^{\Lambda+2}\Phi_\perp^1(\theta,\zeta) = -\Big(\frac{\zeta^2}{2} - \frac{1}{24}\Big)H^0(y)^2\Delta_y U^0(y) \\ + \zeta H^0(y)\big(H^0(y)\Delta_y U^0(y) + \nabla_y H_+^0(y)\cdot\nabla_y U^0(y)\big), \\ \int_\Upsilon \Phi_\perp^1(\theta,\zeta)d\zeta = 0 \end{aligned} \tag{2.17}$$

and remark that $U^1(y,\zeta) = U_\perp^1(y,\zeta) + r^{\Lambda+2}\phi_0^1(\theta)$ still satisfies (2.11) and (2.14).

2.2. **Power-law solutions in dimension $d \geq 3$.** The differential operator on the left hand side of (2.16) reads in spherical coordinates as

$$-A^0 r^{2-d}\partial_r r^d\partial_r - A^0\widetilde{\Delta}_\theta,$$

where $\widetilde{\Delta}_\theta$ is the Laplace-Beltrami operator on the sphere $\mathbb{S}^{d-2} \ni \theta$. It is known, see e.g. [25, Cor. 2.2], that its eigenvalue

$$\widetilde{\mu}_k = k(k+d-3) \ , \quad k \in \mathbb{N}_0 = \{0\}\cup\mathbb{N} \ ,$$

has multiplicity $\varkappa_k$ with

$$\varkappa_0 = 1 \ , \quad \varkappa_p = \frac{(p+d-4)!}{p!(d-3)!}(2p+d-3) \ , \quad p \in \mathbb{N}. \tag{2.18}$$

The corresponding eigenfunctions are obtained by taking traces on $\mathbb{S}^{d-2}$ of homogeneous kth degree harmonic polynomials in $\mathbb{R}^{d-1} \ni y$.

Thus, power-law solutions (2.2) of the equation (2.16) have the following exponents Λ_0, [2]

$$\Lambda_k^{\pm}(d) = \frac{1}{2}\big(1 - d \pm \sqrt{(1-d)^2 + 4k(k+d-3)}\,\big) \ , \quad k \in \mathbb{N}_0. \tag{2.19}$$

Clearly,

$$\Lambda_0^+(d) = 0 \ , \quad \Lambda_0^-(d) = 1 - d < 0 \tag{2.20}$$

and

$$\begin{aligned} &\Lambda_k^{\pm}(3) = \sqrt{k^2+1} - 1 \in (k-1, k) \Rightarrow \\ &\Lambda_1^+(3) = \sqrt{2} - 1 \approx 0.414 \ , \ \Lambda_2^+(3) = \sqrt{5} - 1 \approx 1.236. \end{aligned} \tag{2.21}$$

Furhermore, the functions

$$[3, +\infty) \ni d \mapsto \Lambda_k^+(d)$$

are monotone increasing and

$$\lim_{d \to +\infty} \Lambda_k^+(d) = k. \tag{2.22}$$

Thus, $\lambda_1^+(d) \in (0,1)$ and $\lambda_{1+p}^+(d) > 1$, $p \in \mathbb{N}$, in all dimensions $d \geq 3$. A direct calculation of the Sobolev norms shows that the power-law solutions $U_{+k}^0(y)$ in (2.3) with the exponents $\Lambda_k^+(d)$, $k \in \mathbb{N}_0$, belong to $H^2(\Pi)$ for any $d \geq 3$, although $\nabla_y U_{+1}^0$ is unbounded in Π.

The exponents Λ_k^-, $k \in \mathbb{N}_0$, are negative and belong to $(-\infty, 1-d]$, and consequently the corresponding power-law solutions are not in the space $H^1(\Pi)$.

2.3. **Formal infinite series.** The power-law solution $U^0(y)$ with $\Lambda > 0$ satisfies the problem (1.3)–(1.4) only approximatively, leaving small discrepancies in the equation (1.3) in Π and in the Neumann condition (1.4) in $\varpi_\pm$. To compensate these we construct the formal series

$$U^0(y) + \sum_{j=1}^{\infty} U^j(y,\zeta) = r^\Lambda \Phi^0(\theta) + \sum_{j=1}^{\infty} r^{\Lambda+2j} \Phi^j(\theta,\zeta) \tag{2.23}$$

containing the already chosen main term (2.2) and the infinite asymptotic "tail". We set

$$\Phi^j(\theta,\zeta) = \Phi_0^j(\theta) + \Phi_\perp^j(\theta,\zeta) \ , \quad \int_{-1/2}^{1/2} \Phi_\perp^j(\theta,\zeta) d\zeta = 0 \ ,$$

so that in particular $\Phi_0^0 = \Phi^0$, $\Phi_\perp^0 = 0$ and $\Phi_\perp^1$ is given in (2.17).

By virtue of (2.10), (1.11), (2.12), we obtain the decomposition

$$\begin{aligned} \Delta_x + \lambda &= \frac{1}{H(y)^2} \frac{\partial^2}{\partial \zeta^2} + \Big(\nabla_y - \frac{1}{H(y)} \big(\zeta \nabla_y H(y) + \nabla_y h(y) \big) \partial_\zeta \Big)^2 \\ &= \frac{1}{H^0(y)^2} \Big(\frac{\partial^2}{\partial \zeta^2} + \sum_{j=1}^{\infty} r^{2j} L_j \big(\theta, \zeta, r\frac{\partial}{\partial r}, \widetilde{\nabla}_\theta, \frac{\partial}{\partial \zeta} \big) \Big) \end{aligned} \tag{2.24}$$

[2]For $d = 2$ this formula is used at $k = 0$ only.

where $\widetilde{\nabla}_\theta$ is the spherical part of the gradient operator and the dependence of the differential operators L_j, $j \geq 2$, on the fixed parameter λ is not displayed. Moreover, the normal derivatives (2.13) look as follows:

$$\frac{\partial}{\partial \nu^\pm} = \pm \frac{1}{H^0(y)^2}\Big(\frac{\partial}{\partial \zeta} + \sum_{j=1}^{\infty} r^{2j} N_j^\pm\big(\theta, r\frac{\partial}{\partial r}, \widetilde{\nabla}_\theta, \frac{\partial}{\partial \zeta}\big)\Big). \tag{2.25}$$

We insert the formal series (2.23) and (2.24), (2.25) into the restriction of the problem (1.3), (1.4) to the gap and extract terms of order $r^{\Lambda+2j}$. We obtain an iterative sequence of Neumann problems for ordinary differential equations

$$-\partial_\zeta^2 U^j(y,\zeta) = F^j(y,\zeta)\ , \quad \zeta \in \Upsilon, \quad \pm\partial_\zeta U^j(y, \pm 1/2) = G_\pm^j(y)\ , \tag{2.26}$$

where

$$F^j(y,\zeta) = \sum_{n=1}^{j} r^{2n} L_n\big(\theta, \zeta, r\frac{\partial}{\partial r}, \widetilde{\nabla}_\theta, \frac{\partial}{\partial \zeta}\big) U^{j-n}(y,\zeta)\ ,$$

$$G_\pm^j(y,\zeta) = \sum_{n=1}^{j} r^{2n} N_n^\pm\big(\theta, r\frac{\partial}{\partial r}, \widetilde{\nabla}_\theta, \frac{\partial}{\partial \zeta}\big) U^{j-n}(y,\zeta)\ . \tag{2.27}$$

Noting that

$$\begin{aligned} &r^2 L_1\big(\theta, \zeta, r\partial r, \widetilde{\nabla}_\theta, \partial\zeta\big) \\ &= -H^0(y)^2\big(\nabla_y - H^0(y)^{-1}\big(\zeta\nabla_y H^0(y) + \nabla_y h^0(y)\big)\partial_\zeta\big)^2\ , \\ &r^2 N_1^\pm\big(\theta, r\partial r, \widetilde{\nabla}_\theta, \partial\zeta\big) \\ &= -H^0(y)\nabla_y H_\pm^0(y)\cdot\Big(\nabla_y - H^0(y)^{-1}\big(\frac{1}{2}\nabla_y H^0(y) + \nabla_y h^0(y)\big)\partial_\zeta\Big) \end{aligned} \tag{2.28}$$

we rewrite the compatibility condition

$$\int_\Upsilon F^j(y,\zeta)d\zeta + \sum_\pm G_\pm^j(y) = 0 \tag{2.29}$$

of the problem (2.26) as the inhomogeneous differential equation

$$-\nabla_y\cdot\big(H^0(y)\nabla_y U_0^{j-1}(y)\big) = H^0(y)^{-1}F_0^j(y)\ , \quad y \in \mathbb{R}_\bullet^{d-1}, \tag{2.30}$$

where

$$\begin{aligned} F_0^j &= -\int_\Upsilon \big(r^2 L_1 U_\perp^{j-1} + \sum_{n=2}^{j} r^{2n} L_n U^{j-n}\big)d\zeta \\ &\quad - \sum_\pm \Big(r^2 N_1^\pm U_\perp^{j-1} + \sum_{n=2}^{j} r^{2n} N_n^\pm U^{j-n}\Big)\Big|_{\zeta=\pm 1/2}. \end{aligned} \tag{2.31}$$

The expression

$$-\int_\Upsilon r^2 L_1 U_0^{j-1} d\zeta - \sum_\pm r^2 N_1^\pm U_0^{j-1}\Big|_{\zeta=\pm1/2}$$

is a term of (2.29) that is missing in (2.31), and according to (2.28) it converts into

$$\int_\Upsilon H^0(y)^2\Delta_y U_0^{j-1}(y)d\zeta + \sum_\pm H^0(y)\nabla_y H_\pm^0(y)\cdot\nabla_y U_0^{j-1}(y)$$

$$= H^0(y)\nabla_y \cdot \big(H^0(y)\nabla_y U_0^{j-1}(y)\big),$$

which explains the formula (2.30). Notice that in the opening case $j = 1$ the sum (2.31) is null because $U_\perp^0 = 0$, hence, the equation (2.30) coincides with (2.26) and is thus satisfied.

The next lemma [9], see also [22, Thm. 3.5.6, Lem 3.3.1], yields a solution of the differential equation (2.30) in the general case.

Lemma 2.1. *The equation* (2.30) *with the right hand side*

$$F_0^j(y) = r^{\Lambda+2(j-1)}\Psi_0^j(\theta, \ln r), \tag{2.32}$$

where Ψ_0^j is smooth in the variable $\theta \in \mathbb{S}^{d-1}$ and a polynomial of $\ln r$, has a solution

$$U_0^{j-1}(y) = r^{\Lambda+2(j-1)}\Phi_0^{j-1}(\theta, \ln r), \tag{2.33}$$

where also Φ_0^j has the above mentioned properties of Ψ_0^j. The solution (2.33) *is unique and $deg\,\Phi_0^{j-1} = deg\,\Psi_0^j$, if $\Lambda + 2(j-1)$ does not coincide with any of the exponents* (2.19). ⊠

Remark 2.2. We remark that in the case

$$\Lambda + 2(j-1) = \Lambda_k^+ \tag{2.34}$$

we have $\deg \Phi_0^{j-1} = 1 + \deg \Psi_0^j$ and a general solution of the equation (2.16) becomes

$$U_0^{j-1}(y) + c_j r^{\Lambda_k^+}\Phi_k^+(\theta) \tag{2.35}$$

with arbitrary c_j; note that some terms explicitly depend on $\ln r$. However, in the next section we will prove an algebraic statement, Lemma 2.3, according to which (2.34) never happens, when Λ is of the form (2.19) (with any number in place of k).

The compatibility condition (2.30) is now satisfied and the component $U_\perp^j(y,\zeta) = r^{\Lambda+2j}\Phi_\perp^j(\theta,\zeta)$ of zero mean is determined uniquely from the problem (2.26). The other component $U_0^j(y,\zeta) = r^{\Lambda+2j}\Phi_0^j(\theta,\zeta)$ is not found at the jth step, but Lemma 2.1 gives the component $U_0^{j-1}(y)$ of the previous term $U^{j-1}(y,\zeta)$ in (2.23), and in this way we also find $U_0^j(y)$ while solving the problem (2.26) on the next step with $j \mapsto j+1$.

We emphasize that in our formal procedure we can choose $c_j = 0$ in the general solution (2.35), because we have at hand another formal series of type (2.23) with the initiating power-law solution $r^{\Lambda_k^+}\Phi_k^+(\theta)$ for the compensation of this choice.

Summarizing the above considering and taking into account the proof of the next section, we have shown that every power-law solution $r^\Lambda\Phi^0(\theta)$ of the equation (2.16) gives rise to an infinite asymptotic tail in (2.23) whose coefficients do not explicitly depend on $\ln r$.

2.4. **Non-existence of power-logarithmic solutions.** We complete the study of the case dimension $d \geq 3$ by showing that in our case terms with polynomial dependence on $\ln r$ do not exist, although in general the Kondratiev theory contains that possibility. This is a consequence of the following algebraic observation.

Lemma 2.3. *For $d \geq 3$, the equation*

$$4p + \sqrt{(d-1+2k)^2 - 8k} = \sqrt{(d-1+2q)^2 - 8q} \tag{2.36}$$

does not have a solution with $p, k, q \in \mathbb{N} = \{1, 2, 3, \dots\}$.

By taking a suitable p , this implies that (2.34) does not have a solution, since

$$2\Lambda = 1 - d + \sqrt{(d-1)^2 + 4q(q+d-3)} = 1 - d + \sqrt{(d-1+2q)^2 - 8q}$$

for some q.

Proof. Let us denote in the following $d' = d-1$. We first claim that the expression $(d'+2k)^2 - 8k$ cannot be a square of a positive integer. To prove this we suppose the contrary, so we have

$$(d'+2k)^2 - 8k = (d'+2k-m)^2 \tag{2.37}$$

for some $m \in \mathbb{N}$ (if $m \leq 0$, then $(d'+2k-m)^2 > (d'+2k)^2 - 8k$). Cancelling the same terms on both sides, (2.37) is equivalent to

$$m^2 - 4km - 2d'm = -8k \tag{2.38}$$

so that m must be even, $m = 2t$ for some positive integer t, and (2.38) is equivalent to

$$t^2 - 2kt - d't = -2k \quad \Leftrightarrow \quad t = 2k + d' - \frac{2k}{t}. \tag{2.39}$$

It is easy to see that no $t \in \mathbb{N}$ can solve this. First, if $t \geq k$, then the term $2k/t$ can be integer only in the cases $t = 2k$ or $t = k$, but none of these solves (2.39) for $k, d \in \mathbb{N}$, $d \geq 3$. If $t < k$, $t \in \mathbb{N}$, is a solution of (2.39), then we must have $nt = 2k$ for some $n \in \mathbb{N}$ in order to make $2k/t$ into an integer. But then (2.39) becomes equivalent with

$$t = nt + d' - n \quad \Leftrightarrow \quad t = \frac{n-d'}{n-1}$$

which does not have a solution $t, n, d' \in \mathbb{N}$, $d' \geq 2$. This proves our first claim.

Let us still denote $a = (d-1+2k)^2 - 8k$ and $b = (d-1+2q)^2 - 8q$. Then, by squaring, (2.36) is equivalent with

$$2\sqrt{ab} = a + b - 16p^2. \tag{2.40}$$

Writing $b = a\ell^2$ for some real number $\ell > 0$, we have $\ell^2 = b/a \in \mathbb{Q}$, the set of rational numbers. Moreover, (2.40) is equivalent with

$$2a\ell = a(1+\ell^2) - 16p^2 \quad \Leftrightarrow \quad \ell = \frac{1}{2a}\big(a(1+\ell^2) - 16p^2\big)$$

where the right hand side is rational, hence $\ell \in \mathbb{Q}$. Finally,

$$2a\ell = a(1+\ell^2) - 16p^2 \quad \Leftrightarrow \quad a(\ell^2 - 2\ell + 1) = 16p^2 \quad \Leftrightarrow \quad a = \frac{16p^2}{(\ell-1)^2} =: w^2.$$

where $w \in \mathbb{Q}$. Writing w as the quotient with prime numbers p_j and q_j different from each other, we get

$$w = \frac{p_1 \dots p_n}{q_1 \dots q_m} \quad \Rightarrow \quad a = \frac{p_1^2 \dots p_n^2}{q_1^2 \dots q_m^2}$$

We see that all the factors q_j in the denominator must be equal to one, since a is an integer. But $a = (p_1 \dots p_n)^2$ contradicts with the claim proven in the beginning. ⊠

2.5. **The planar case.** In dimension $d = 2$ the gap Π in (1.7) disintegrates into two cuspidal domains $\Pi_\pm = \{x = (y, z) \in \Pi : \pm y > 0\}$, and accordingly, the limit equation (2.16) is posed on the separated semi-axes $\mathbb{R}_\pm$. In the sequel we deal with the right cusp Π_+ (shaded in Fig. 1.1.b)) and the Euler type ordinary differential equation

$$-A^0 \frac{d}{dy} y^2 \frac{d}{dy} U^0(y) = 0 \quad , \quad y \in \mathbb{R}_+ = (0, +\infty) \ . \tag{2.41}$$

As for its solutions,

$$U^0_+(y) = 1 \text{ for } \Lambda^0_+ = 0 \text{ and } U^0_-(y) = y^{-1} \text{ for } \Lambda^0_- = -1, \tag{2.42}$$

the second one does not belong to $H^1(\Pi_+)$. The first, the constant function C_0, satisfies the Neumann condition but leaves the discrepancy λC_0 in the Helmholtz equation. To compensate it, we construct the asymptotic tail in

$$C_0 + \sum_{k=1}^{\infty} U^k(y, \zeta) = C_0 + \sum_{k=1}^{\infty} y^{2k} \Phi^k(\zeta). \tag{2.43}$$

To compensate the above mentioned discrepancy, we set

$$U^1(y, \zeta) = \frac{1}{2} C_1 y^2 \tag{2.44}$$

and specify the Neumann problem (2.26) with $j = 2$ as follows:

$$\begin{aligned} -\frac{\partial^2}{\partial \zeta^2} U^2(y, \zeta) &= H^0(y)^2 (\lambda C^0 + C^1_1) \ , \quad \zeta \in \big(-\frac{1}{2}, \frac{1}{2}\big) \ , \\ -\frac{\partial}{\partial \zeta} U^2\big(y, \pm\frac{1}{2}\big) &= H^0(y) \partial_y H^0_\pm(y) C_1 y \ . \end{aligned} \tag{2.45}$$

The compatibility condition for this problem is simply obtained from the Newton-Leibnitz-formula

$$\int_{-1/2}^{1/2} \frac{\partial^2}{\partial \zeta^2} U^2(y, \zeta) d\zeta = \frac{\partial U^2}{\partial \zeta}(y, \frac{1}{2}) - \frac{\partial U^2}{\partial \zeta}(y, -\frac{1}{2}),$$

and taking into account (2.12) it reduces to the relation

$$C_1 = -\frac{1}{3} \lambda C_0 \ .$$

Hence, the solution of (2.45) reads as

$$U^2(y, \zeta) = -\frac{1}{3} H^0(y) \lambda C_0 \big(H^0(y) \zeta^2 + 2h^0(y) \zeta \big) + C_2 y^4 \ , \tag{2.46}$$

where the coefficient C_2 is not fixed yet. In view of (2.8), (1.11), and (2.12), the function (2.46) is quadratic in z and smooth in $y \geq 0$, i.e., it belongs to $C^\infty\big(\overline{\Pi_+}\big)$.

In the next lemma we will prove that Φ^k is a polynomial of degree $\leq k$ in ζ,

$$\Phi^k(\zeta) = b_k \zeta^k + \varphi^k(\zeta) \tag{2.47}$$

with $\deg \varphi^k \leq k - 1$. This implies that each term of the tail in (2.43) belongs to $C^\infty\big(\overline{\Pi_+}\big)$ so that after the justification of asymptotic formulas for eigenfunctions u_n of the problem (1.3), (1.4) in Section 3, we may conclude that $u_n \in C^\infty\big(\overline{\Pi_\pm}\big)$. However, the limit values $u_n(\pm 0, 0)$, of course, can differ from each other due to the disconnectedness of the gap $\Pi \subset \mathbb{R}^2$. Thus, the eigenfunction u_n is not differentiable at the point $\mathcal{O} \subset \overline{\Omega}$: u_n and its derivatives can have jumps as x approaches $\mathcal{O}$ from the right and left in the domain Ω. However, Remark 1.2 shows that some eigenfunctions belong to $C^\infty(\overline{\Omega})$, and also by the considerations of this section, the equality $u_n(\mathcal{O}) = 0$ implies that $u_n \in C^\infty(\overline{\Omega})$, since $C_0 = 0$ and all terms in (2.43) vanish.

Lemma 2.4. *Formula* (2.47) *is valid for all* $k \in \mathbb{N}$.

Proof. Proceeding by induction, the cases $k = 1, 2$ follow from (2.44) and (2.46). Let us assume that (2.47) holds for $k \leq j-1$. According to (2.10), the differential operators L_p in (2.24) are at most quadratic in ∂_ζ and $\zeta\partial_\zeta$, and hence, considering polynomials of the variable ζ, we have

$$\deg L_n U^{j-n} \leq \deg \Phi^{j-n} \ , \quad n = 1, \dots, j.$$

The right hand side F^j, (2.27), of the differential equation (2.26) satisfies

$$F^j = y^2 L_1 U^{j-1} + f^j \ , \quad \deg f^j \leq j-2. \tag{2.48}$$

Moreover, in dimension $d = 2$ the first formula (2.28) reads as

$$r^2 L_1(\zeta, y\partial_y, \partial_\zeta) = -y^4 \big(A^0 \partial_y - 2\frac{1}{y}(A^0\zeta + a^0)\partial_\zeta\big)^2,$$

where A^0 and a^0 are taken from (2.12). Then, we write

$$\begin{aligned} & r^2 L_1(\zeta, y\partial_y, \partial_\zeta) \\ &= -y^4 \big(A^0\partial_y - 2\frac{1}{y}(A^0\zeta + a^0)\partial_\zeta\big)\big(2(j-1)A^0\zeta - 2(j-1)(A^0\zeta + a^0)\big) y^{2j-3}\zeta^{j-2} \\ &= 2(j-1)a^0 y^4 \big(A^0\partial_y - 2\frac{1}{y}(A^0\zeta + a^0)\partial_\zeta\big) y^{2j-3}\zeta^{j-2} \end{aligned}$$

and notice that the degree of this polynomial of ζ is nothing but $j-2$. Thus, $\deg F^j \leq j-2$ in (2.48) and formula (2.47) follows easily .

3. Justification of asymptotic expansions.

The justification scheme presented here consists of several steps. First, we formulate the Kondratiev theorem on asymptotics, related to the equation (2.16). Second, we use a novel approach, which is much more simple than in [20, 24], to reduce the original problem in Π to the limit problem in $\mathbb{R}^{d-1}_\bullet$ and derive the one-term asymptotic formula for eigenfunctions. Then, we iterate this result, and in combination with the formal procedure of Section 2 obtain expansions with remainders of any prescribed power-law decay rate as $x \to \mathcal{O}$. Finally, we produce weighted Hölder estimates and make the desired conclusions on the smoothness properties of eigenfunctions in dimensions 2 and 3.

3.1. **Basics of the Kondratiev theory.** Let $V^1_\beta(\mathbb{R}^{d-1})$ be the weighted Sobolev space defined as the completion of $C^\infty_0(\mathbb{R}^{d-1}_\bullet)$ (the space of infinitely differentiable and compactly supported functions) with respect to the norm

$$\|\mathbf{u}; V^1_\beta(\mathbb{R}^{d-1})\| = \Big(\|r^\beta \nabla_y \mathbf{u}; L^2(\mathbb{R}^{d-1})\|^2 + \|r^{\beta-1}\mathbf{u}; L^2(\mathbb{R}^{d-1})\|^2\Big)^{1/2},$$

where $\beta \in \mathbb{R}$ is a weight index. The relations

$$\mathbf{u} \in V^1_\beta(\mathbb{R}^{d-1}) \ \ , \ \ \widetilde{\mathbf{u}} \in V^1_\gamma(\mathbb{R}^{d-1}) \ \ , \ \ \beta > \gamma \tag{3.1}$$

imply lower singularity or faster decay for the function $\widetilde{\mathbf{u}}(y)$ than for $\mathbf{u}(y)$ as $r \to +0$.

We reformulate the inhomogeneous equation (2.16) for the function $\mathbf{u} \in V^1_{1+\beta}(\mathbb{R}^{d-1})$ as the integral identity

$$(H_0 \nabla_y \mathbf{u}, \nabla_y \mathbf{v})_{\mathbb{R}^{d-1}} = \mathbf{f}(\mathbf{v}) \quad \forall\, \mathbf{v} \in V^1_{1-\beta}(\mathbb{R}^{d-1}) \ , \tag{3.2}$$

where $\mathbf{f} \in V^1_{1-\beta}(\mathbb{R}^{d-1})^*$ is a linear functional in $V^1_{1-\beta}(\mathbb{R}^{d-1})$,

$$\mathbf{f}(\mathbf{v}) = (\mathbf{f}_0, \mathbf{v})_{\mathbb{R}^{d-1}} - \sum_{n=1}^{d-1} \Big(\mathbf{f}_n, \frac{\partial \mathbf{v}}{\partial y_n}\Big)_{\mathbb{R}^{d-1}} , \quad \mathbf{f}_0 \in L^2_\beta(\mathbb{R}^{d-1}),\ \mathbf{f}_n \in L^2_{\beta-1}(\mathbb{R}^{d-1}), \tag{3.3}$$

where the norm of the weighted Lebesgue space $L^2_\beta(\mathbb{R}^{d-1})$ equals $\|r^\beta \mathbf{f}_0; L^2(\mathbb{R}^{d-1})\|$. Note that the inner product $(\ ,\ )_{\mathbb{R}^{d-1}}$ of $L^2(\mathbb{R}^{d-1})$ is extended to the dual pairing of appropriate weighted Lebesgue spaces so that all terms in (3.2) and (3.3) are properly defined. The integral identity is obtained by writing the equation (2.30) for the unknown function $\mathbf{u}$ and the right hand side

$$F = \mathbf{f}_0 + \sum_{n=1}^{d-1} \frac{\partial \mathbf{f}_n}{\partial y_n} \ ,$$

and, as usual, multiplying it by a test function $\mathbf{v}$ and integrating by parts, cf. [13].

The following assertion originates in [9] and can also be found in [16, Thm. 3.5.6], [10, Thm. 8.3.3]. It contains uncommon restrictions for the weight indices which happen to be very convenient for our purposes. In the sequel we will need the indices

$$\beta < (d-3)/2 \tag{3.4}$$

which will exclude power-law solutions with negative exponents (2.19); however, we do not yet pose the restriction (3.4) in this section.

Theorem 3.1. *Let β and γ be as in* (3.1) *and let $\mathbf{u} \in V_{1+\beta}(\mathbb{R}^{d-1})$ be a solution of the problem* (3.2) *with the right hand side*

$$\mathbf{f} \in V^1_{1-\beta}(\mathbb{R}^{d-1}) \cap V^1_{1-\gamma}(\mathbb{R}^{d-1}).$$

We also assume that the endpoints of the interval

$$\upsilon = \Big(\frac{1}{2}(1-d) - \beta, \frac{1}{2}(1-d) - \gamma\Big) \tag{3.5}$$

do not coincide with any of the exponents (2.19).[3]

1°. *If there are no exponents* (2.19) *in the interval* (3.5), *then $\mathbf{u} \in V^1_{1-\gamma}(\mathbb{R}^{d-1}) \cap V^1_{1-\beta}(\mathbb{R}^{d-1})$.*

2°. *Assuming that υ contains only one of the exponents in* (2.19), *we denote it by Λ and obtain the other solution*

$$\widetilde{\mathbf{u}} = \mathbf{u} - \sum_{p=1}^{\varkappa} b_p r^\Lambda \Phi_p(\theta) \in V^1_{1+\gamma}(\mathbb{R}^d) \tag{3.6}$$

for the problem (3.2) *with the new weight index γ. In* (3.6), *the functions $r^\Lambda\Phi_1, \dots,$ $r^\Lambda \Phi_\varkappa$ form a basis of the linear space of power-law solutions with the exponent Λ,*

[3]For $d = 2$, only the exponent $\Lambda^0_+ = 0$ can be in question.

cf. (2.2)*, and its dimension* $\varkappa$ *is given by* (2.18)*. The coefficients* $b_1, \dots, b_\varkappa$ *depend on* $\mathbf{u}, \mathbf{f}$ *and satisfy the estimate*

$$|b_1| + \ldots + |b_\varkappa| \le c_{\beta\gamma}\big(\|\mathbf{f}; V^1_{1-\beta}(\mathbb{R}^{d-1})^*\| + \|\mathbf{f}; V^1_{1-\gamma}(\mathbb{R}^{d-1})^*\|\big). \ \boxtimes$$

Formula (3.6) can be regarded as the asymptotics of the solution $\mathbf{u}$ with the remainder $\widetilde{\mathbf{u}}$.

3.2. **First result on asymptotics in Ω.** We denote by $\mathcal{V}^1_\beta(\Pi)$ the weighted Sobolev space in the gap Π, endowed with the norm

$$\|u; \mathcal{V}^1_\beta(\Pi)\| = \big(\|\nabla_x u; \mathcal{L}^2_\beta(\Pi)\|^2 + \|u; \mathcal{L}^2_{\beta-1}(\Pi)\|^2\big)^{1/2},$$

where $\mathcal{L}^2_\beta(\Pi)$ is the weighted Lebesgue space with norm $\|f; \mathcal{L}^2_\beta(\Pi)\| = \|r^\beta f; L^2(\Pi)\|$. Let $u \in \mathcal{V}^1_\beta(\Pi)$ be either the eigenfunction u_n or the remainder $\widetilde{u}_n$ in its asymptotic representation multiplied by a proper cut-off-function, and assume that it is smooth in $\overline{\Pi} \setminus \mathcal{O}$, vanishes for $|y| > R$ and satisfies the problem

$$-\Delta_x u(x) - \lambda u(x) = f(x),\ x \in \Pi \ \ , \ \ \partial_{\nu_\pm} u(x) = g_\pm(y),\ x \in \varpi_\pm \tag{3.7}$$

with the right hand sides

$$f \in \mathcal{L}^2_{\beta+1-\delta}(\Pi)\ , \quad g_\pm \in \mathcal{L}^2_{\beta-\delta}(\varpi_\pm)\ , \quad \delta > 0. \tag{3.8}$$

The problem (3.7) corresponds to the integral identity [13]

$$(\nabla_x u, \nabla_x v)_\Pi - \lambda(u, v)_\pi = (f, v)_\Pi + \sum_\pm (g_\pm, v)_{\varpi_\pm}, \tag{3.9}$$

where test functions belong to the space $C^\infty_c(\overline{\Pi} \setminus \mathcal{O})$; by a completion argument, we can take any $v \in \mathcal{V}^1_{-\beta}(\Pi)$. However, we choose $v(y, z) = \mathbf{v}(y)$ to be independent of z.

Let us show that the functions $\mathbf{u}$ and $u_\perp$ introduced in (2.1) and (2.3) have the desired properties.

Lemma 3.2. *The function* $\mathbf{u}$ *belongs to* $V^1_{1+\beta}(\mathbf{B})$ *and*

$$\|\mathbf{u}; V^1_{1+\beta}(\mathbf{B})\| \le c\|u; \mathcal{V}^1_\beta(\Pi)\|.$$

Proof. First of all, we have

$$\|r^\beta \mathbf{u}; L^2(\mathbf{B})\|^2 = \int_{\mathbf{B}} r^{2\beta} H(y)^{-2} \Big| \int_{-H_-(y)}^{H_+(y)} u(y, z)dz \Big|^2 dy$$

$$\le \int_\Pi r^{2\beta} H(y)^{-1} |u(x)|^2 dx \le c_H \int_\Pi r^{2(\beta-1)} |u(x)|^2 dx \le c\|u; \mathcal{V}^1_\beta(\Pi)\|^2. \tag{3.10}$$

Furthermore, according to definition (2.1)

$$\nabla_y \mathbf{u}(y) = -\frac{\nabla_y H(y)}{H(y)} \mathbf{u}(y)$$

$$+ \frac{1}{H(y)} \int_{-H_-(y)}^{H_+(y)} \nabla_y u(y, z)dz + \sum_\pm \frac{\nabla_y H(y)}{H(y)} u^\pm(y), \tag{3.11}$$

where $u^\pm = u\big|_{\varpi_\pm}$. The relation (3.10) with obvious changes proves that the first two terms on the right hand side of (3.11) belong to $L^2_{1+\beta}(\mathbf{B})$ with the corresponding norm estimates. The inequality $|\nabla_y H_\pm(y)|H(y)^{-1} \le c_H r^{-1}$ and the following calculation finish the proof:

$$\sum_\pm \int_{\mathbf{B}} r^{2\beta}|u^\pm(y)|^2 dy = 2\int_{\mathbf{B}} r^{2\beta} \int_{-H_-(y)}^{H_+(y)} \frac{\partial}{\partial z}\big(\zeta u(y,z)^2\big) dz dy$$

$$\le c\int_\Pi r^{2\beta}\Big(\frac{1}{H(y)}|u(x)| + |\partial_z u(x)|\Big)|u(x)|dx$$

$$\le \int_\Pi r^{2\beta}\big(r^{-2}|u(x)|^2 + |\partial_z u(x)|^2\big)dx \le c\|u;\mathcal{V}^1_\beta(\Pi)\|^2. \quad \boxtimes \tag{3.12}$$

Lemma 3.3. *We have $u_\perp \in \mathcal{L}^2_{\beta-2}(\Pi)$ and $u^\pm_\perp = u_\perp|_{\varpi_\pm} \in \mathcal{L}^2_{\beta-1}(\varpi_\pm)$, and there holds the estimate*

$$\|u_\perp;\mathcal{L}^2_{\beta-2}(\Pi)\| + \|u^\pm_\perp;\mathcal{L}^2_{\beta-1}(\varpi_\pm)\| \le c\|u;\mathcal{V}^1_\beta(\Pi)\|.$$

Proof. The orthogonality condition (2.4) and the Poincaré inequality yield

$$\int_\Pi r^{2\beta}H(y)^{-2}|u_\perp(y,z)|^2 dydz \le \frac{1}{\pi^2}\int r^{2\beta}\Big|\frac{\partial u}{\partial z}(x)\Big|^2 dx \le \frac{1}{\pi^2}\|u;\mathcal{V}^1_\beta(\Pi)\|^2.$$

Using the calculation (3.12), the Newton-Leibnitz formula, and the definition (2.8) of ζ, we complete the proof by

$$\sum_\pm \int_{\mathbf{B}} r^{2\beta-2}|u^\pm_\perp(y)|^2 dy = 2\int_{\mathbf{B}} r^{2\beta-2} \int_{-H_-(y)}^{H_+(y)} \frac{\partial}{\partial z}\big(\zeta u_\perp(y,z)^2\big) dz dy$$

$$\le c\int_\Pi r^{2\beta}\Big(\frac{1}{H(y)}|u_\perp(x)| + |\partial_z u_\perp(x)|\Big)|u(x)|dx$$

$$\le c\int_\Pi r^{2\beta}\big(r^{-4}|u_\perp(x)|^2 + |\partial_z u(x)|^2\big)dx. \quad \boxtimes$$

We observe that

$$\nabla_y \int_{-H_-(y)}^{H_+(y)} u(y,z)dz = \int_{-H_-(y)}^{H_+(y)} \nabla_y u(y,z)dz + \sum_\pm u\big(y,\pm H_\pm(y)\big)\nabla_y H_\pm(y)$$

and therefore

$$\int_{-H_-(y)}^{H_+(y)} \nabla_y u(y,z)dz = \nabla_y\big(H(y)\mathbf{u}(y)\big) - \mathbf{u}(y)\nabla_y H(y) - \sum_\pm u^\pm_\perp(y)\nabla_y H_\pm(y).$$

As a result, the identity (3.9) with $v(y,z) = \mathbf{v}(y)$ becomes

$$\big(H\nabla_y \mathbf{u}, \nabla_y \mathbf{v}\big)_{\mathbf{B}} - \lambda(H\mathbf{u},\mathbf{v})_{\mathbf{B}}$$

$$(3.13)\qquad = \Big(\int\limits_{H_-}^{H_+} f dz, \mathbf{v}\Big)_{\mathbf{B}} + \sum_{\pm} \big(J_\pm g_\pm, \mathbf{v}\big)_{\mathbf{B}} + \sum_{\pm} \big(J_\pm u_\perp^\pm \nabla_y H_\pm, \nabla_y \mathbf{v}\big)_{\mathbf{B}},$$

where $ds_\pm = J_\pm(y)dy$, see (2.13), is the area element of $\varpi_\pm$.

Since u was assumed to vanish near the vertical side $\{x \in \partial\Pi : |y| = R\}$ of the gap (1.7), the mean value function (2.1) does the same near the circle $\partial\mathbf{B}$. Thus, we can interprete (3.13) as the integral identity (3.2) with any test function $\mathbf{v} \in C_c^\infty(\mathbb{R}^{d-1}_\bullet)$ and the right hand side (3.2) with

$$\mathbf{f}_0(y) = \int\limits_{-H_-(y)}^{H_+(y)} f(y,z)dz + \lambda H(y)\mathbf{u}(y) - \sum_{\pm} J_\pm(y) g_\pm(y)\ ,$$

$$\mathbf{f}_n(y) = \big(H_0(y) - H(y)\big)\frac{\partial \mathbf{u}}{\partial y_n}(y) + \sum_{\pm} J_\pm(y) u_\perp^\pm(y) \frac{\partial H_\pm}{\partial y_n}(y)\ , n = 1, \dots, d-1.$$

Moreover, $\mathbf{f}_0$, $\mathbf{f}_n$ vanish outside the disc $\mathbf{B}$ and

$$(3.14)\qquad \mathbf{f}_0 \in L^2_\gamma(\mathbf{B})\ ,\ \mathbf{f}_n \in L^2_{\gamma-1}(\mathbf{B}) \text{ for all } \gamma \geq \max\{\beta - \delta, \beta - 2\}.$$

To get this restriction on the new weight index γ, we took into account the inequalities $|H(y) - H_0(y)| \leq cr^4$ and $|\nabla_y H_\pm(y)| \leq cr^2$, $y \in \mathbf{B}$, the inclusions in Lemmas 3.2 and 3.3 as well as $H^{-1}\int f dz \in L_{\beta+2-\delta}(\mathbf{B})$ following from the assumption (3.8) and a modified estimate (3.10).

Fixing $\gamma \geq \beta - \min\{1, \delta\}$, cf. (3.14), and assuming that β and γ satisfy the hypothesis in Theorem 3.1, we obtain the representation (3.6), rewritten in the form

$$(3.15)\qquad \mathbf{u}(y) = \chi(r) \sum_{p=1}^{\varkappa} b_p r^\Lambda \Phi_p(\theta) + \widetilde{\mathbf{u}}(y),$$

where χ is a smooth cut-off function such that $0 \leq \chi \leq 1$ and

$$(3.16)\qquad \chi(r) = 0 \text{ for } r \geq 2R/3\ ,\quad \chi(r) = 1 \text{ for } r \leq R/3.$$

Notice that the remainder $\widetilde{\mathbf{u}}$ belongs to $V^1_{1+\gamma}(\mathbf{B})$ and that the sum of the power solutions $U^0_p(y) = r^\Lambda \Phi_p(\theta)$ is defined as zero, if the interval (3.5) does not contain any exponent Λ in (2.19).

With the same convention on the summation, we set

$$(3.17)\qquad u(x) = \mathbf{U}(x) + \widetilde{u}(x)\ ,\quad \mathbf{U}(x) = \chi(x) \sum_{p=1}^{\varkappa} b_p \big(U^0_p(y) + U^1_{p\perp}(y,\zeta)\big),$$

where $U^1_{p\perp}$ is constructed from U^0_p according to (2.17) and $b_1, \dots, b_\varkappa$ come from (3.15). Since $u_\perp \in \mathcal{L}^2_{\beta-2}(\Pi) \subset L^2_{\gamma-1}(\Pi)$ (recall that $\gamma \geq \beta - 1$) and $\chi U^1_{p\perp} \in \mathcal{L}^2_{\gamma-1}(\Pi)$ ($\Lambda \in \upsilon$ and the exponent $\Lambda+2$ in (2.17) is bigger than $\frac{1}{2}(1-d)-\gamma$), the representation with $\widetilde{\mathbf{u}} \in \mathcal{L}^2_{\gamma-1}(\Pi)$ yields

$$(3.18)\qquad \widetilde{u} \in \mathcal{L}^2_{\gamma-1}(\Pi) \text{ and moreover } \widetilde{u}^\pm = \widetilde{u}|_{\varpi_\pm} \in \mathcal{L}^2_\gamma(\varpi_\pm).$$

The last inclusions are derived by a similar argument using Lemmas 3.2 and 3.3. However, we cannot conclude at the moment that $\widetilde{u} \in \mathcal{V}^1_\gamma(\Pi)$ because of insufficient information on the gradient $\nabla_x u_\perp$.

Remark 3.4. At the first glance the functions $U^1_{p\perp}$ seem to be "surplus" terms in (3.17), because

$$\gamma \geq \beta - 1,\ \Lambda \in \upsilon \quad \Rightarrow \quad \Lambda > \frac{1}{2}(1-d) - \beta \geq \frac{1}{2}(1-d) - \gamma - 1\ ,$$
$$\nabla_y U^1_{p\perp}(y,\zeta) = O(r^{\Lambda+1}),\ \partial_z U^1_{p\perp}(y,\zeta) = O(r^{\Lambda}) \Rightarrow \nabla_x U^1_{p\perp} \in \mathcal{L}^2_\gamma(\Pi).$$

However, $\partial^2_z U^1_{p\perp}(y,\zeta) = O(r^{\Lambda-2})$, $\Delta_y U^0_p(y) = O(r^{\Lambda-2})$ and thus we would not be able to obtain "good" properties for the right hand sides in the problem for $\widetilde{u}$ without having $U^1_{p\perp}$ in (3.17).

Theorem 3.5. *Assuming the above mentioned conditions, the representation* (3.17) *holds true with* $\widetilde{u} \in \mathcal{V}^1_\gamma(\Pi)$.

Proof. Recalling the asymptotic procedure in Section 2.3, we see that $\Delta_x \mathbf{U} \in \mathcal{L}^2_{\gamma+1}(\Pi)$ and $\partial_{\nu^\pm}\mathbf{U} \in \mathcal{L}^2_\gamma(\varpi_\pm)$. We compose the following problem for $\widetilde{u} \in \mathcal{V}^1_\beta(\Pi)$:

$$-\Delta_x \widetilde{u} + tr^{-2}\widetilde{u} = \widetilde{f} := f + \lambda u + (\Delta_x + \lambda)\mathbf{U} + tr^{-2}\widetilde{u} \in \mathcal{L}^0_{\gamma+1}(\Pi),$$
$$\partial_{\nu^\pm}\widetilde{u} = \widetilde{g}_\pm := g_\pm - \partial_{\nu^\pm}\mathbf{U} \in \mathcal{L}^0_\gamma(\varpi_\pm), \tag{3.19}$$

where $t > 0$ will be fixed later. Because of the cut-off function (3.16) in (3.17), the function $\widetilde{u}$ vanishes near the surface $\{x \in \Pi : |y| = R\}$ (this property of u has been assumed) and, therefore, we obtain the integral identity

$$(\nabla_x, \nabla_x v)_\Pi + t(r^{-2}\widetilde{u}, v)_\Pi = (\widetilde{f}, v)_\Pi + \sum_\pm (\widetilde{g}_\pm, v)_{\varpi_\pm} \quad \forall\ v \in \mathcal{V}^1_{-\beta}(\Pi), \tag{3.20}$$

cf. (3.14). Aiming to show that $\widetilde{u} \in \mathcal{V}^1_\gamma(\Pi)$, we proceed in the same way as in the proof of Proposition 1.1 but with the following weight function instead of (1.9):

$$\mathcal{R}_\varrho(x) = \begin{cases} r^\gamma & \text{for } R > |y| > \varrho, \\ \varrho^{\gamma-\beta} r^\beta & \text{for } |y| < \varrho. \end{cases} \tag{3.21}$$

Noting that, for any $\varrho \in (0,R)$, $\mathcal{U} = \mathcal{R}_\varrho \widetilde{u} \in \mathcal{V}^1_0(\Pi)$ and $v = \mathcal{R}_\varrho \mathcal{U} \in \mathcal{V}^1_{-\beta}(\Pi)$, we insert the latter into (3.20) and obtain similarly to (1.10)

$$\begin{aligned} &\|\nabla_x \mathcal{U}; L^2(\Pi)\|^2 - \|\mathcal{U}\mathcal{R}^{-1}_\varrho \nabla_x \mathcal{R}_\varrho; L^2(\Pi)\|^2 + t\|r^{-1}\mathcal{U}; L^2(\Pi)\|^2 \\ &= (\mathcal{R}_\varrho \widetilde{f}, \mathcal{R}_\varrho \widetilde{u})_\Pi + \sum_\pm (\mathcal{R}_\varrho \widetilde{g}_\pm, \mathcal{R}_\varrho \widetilde{u})_{\varpi_\pm}. \end{aligned} \tag{3.22}$$

In view of the inclusions (3.18), (3.19) and the inequality $|\mathcal{R}_\varrho(x)| \leq r^\gamma$ (following from (3.21) and $\gamma < \beta$) we conclude that the right hand side of (3.22) is uniformly bounded with respect to $\varrho \in (0,R)$. Since $|\nabla_x \mathcal{R}_\varrho(x)| \leq \max\{|\gamma|, |\beta|\} r^{-1}\mathcal{R}_\varrho(x)$, choosing t such that $t > 1 + 2\max\{\beta^2, \gamma^2\}$ makes the left hand side of (3.22) to exceed

$$\begin{aligned} &\|\nabla_x(\mathcal{R}_\varrho \widetilde{u}); L^2(\Pi)\|^2 + (1 + \max\{|\beta^l, \Gamma^2\})\|r^{-1}\mathcal{R}_\varrho \widetilde{u}; L^2(\Pi)\|^2 \\ \geq\ &\|\mathcal{R}_\varrho \nabla_x \widetilde{u}; L^2(\Pi)\|^2 + \|r^{-1}\mathcal{R}_\varrho \widetilde{u}; L^2(\Pi)\|^2. \end{aligned}$$

Since $\mathcal{R}_\varrho(x)$ is monotone increasing when $\varrho \to +0$, the limit of the last, bounded, expression exists and equals $\|\widetilde{u}; \mathcal{V}^1_\gamma(\Pi)\|^2$. ⊠

3.3. **Asymptotics in weighted Sobolev space.** Let $u_n \in H^1(\Omega) \cap C^\infty(\overline{\Omega} \setminus \mathcal{O})$ be an eigenfunction of the problem (1.3)–(1.4). We multiply it by the cut-off function (3.16) and obtain, for the function $u = \chi u_n$, the inhomogeneous problem (3.7) with smooth right hand sides f and $g_\pm$, which vanish, if $r \notin [R/3, 2R/3]$.

First of all, we deal with the case $d = 2$. Observing that $u = \chi u_n \in H^1(\Pi) \subset \mathcal{V}^1_1(\Pi)$, we first set $\beta = 1$, $\gamma = 0$ and find the negative exponent $\Lambda^0_- = -1$ in the interval $\upsilon = (-3/2, -1/2)$, see (2.42) and (3.5). However, $c_\pm |y|^{-1}$ belongs neither to $H^1(\Pi_\pm)$ nor to $\mathcal{L}^2_{-1}(\Pi_\pm)$, and thus it does not appear in the asymptotics of $u \in H^1(\Pi_\pm)$. Hence, $u \in \mathcal{V}^1_0(\Pi)$, and we may take $\beta = 0$, $\gamma = -1$. The corresponding interval $\upsilon = (-1/2, 1/2)$ includes the second exponent $\Lambda^0_+ = 0$ in (2.42). Theorem 3.5 yields the formula

$$u - \sum_\pm \chi_\pm (C_0^\pm + U_\perp^{1\pm}) \in \mathcal{V}^1_{-1}(\Pi_\pm) \ , \tag{3.23}$$

and in particular determines the constants $C_0^\pm = u(\pm 0, 0)$, which are the first terms in the asymptotic tails (2.43). (Recall that in Section 2.5 we only considered the right half Π_+, Fig. 1.1.b), of the gap Π, but (3.23) makes sense due to the signs $\pm$ and the cut-off functions $\chi_\pm$.) Using the entire series (2.43), we fix some $N \in \mathbb{N}$ and set

$$\widetilde{u} = u - \sum_\pm \chi_\pm \Big(u(\pm 0, 0) + \sum_{p=1}^N U^{p\pm} + U_\perp^{N\pm} \Big) \ , \tag{3.24}$$

although at the moment we only know that $\widetilde{u} \in \mathcal{V}^1_{-1}(\Pi)$. However, the asymptotic procedure in Section 2.3 shows that in the problem (3.7) for $\widetilde{u}$, the right hand sides $\widetilde{f}$ and $\widetilde{g}_\pm$ satisfy

$$\widetilde{f} \in \mathcal{L}^2_{\gamma+1}(\Pi) \ \ , \ \ \widetilde{g}_\pm \in \mathcal{L}^2_\gamma(\varpi_\pm) \quad \text{for all } \gamma \geq -2N - 1/2 \ . \tag{3.25}$$

Since the limit equation (2.41) does not have power solutions (2.2) with positive exponents Λ, we apply Theorem 3.5 $N-1$ times: at each step we diminish the weight index β in the relation $\widetilde{u} \in \mathcal{V}^1_\beta(\Pi)$ by 1, so that we finally obtain $\widetilde{u} \in \mathcal{V}^1_{-N}(\Pi)$. Finally, we choose some $\gamma \in (-N - 1/2, -N)$ so that (3.25) and Theorem 3.6 yield the following assertion.

Theorem 3.6. *For all eigenfunctions u_n, $n \in \mathbb{N}$, of the problem* (1.3), (1.4), *the asymptotic formula* (3.24) *holds with the above constructed sum* (2.43) *and with $\widetilde{u} \in \mathcal{V}^1_\gamma(\Omega)$, where $\gamma \in (-N - 1/2, -N)$ is arbitrary.*

The same scheme applies in dimension $d \geq 3$, although it becomes somewhat cumbersome. We again take some $N \in \mathbb{N}$ and derive an asymptotic form for the eigenfunction u_n with remainder

$$\widetilde{u}_n^N \in \mathcal{V}^1_\sigma(\Omega) \ \text{ with } \sigma = -2N - \delta + (1-d)/2 < 0 \text{ and a small } \delta > 0. \tag{3.26}$$

Notice that by the calculations (2.21)–(2.22), the power-law solutions with exponents Λ_k^+, $k = 0, \ldots, N$, see (2.19), do not belong to $\mathcal{V}^1_0(\Pi)$ but with Λ_k^+, $k > N$, they do.

First of all, we have $u = \chi u_n \in H^1(\Omega) \subset \mathcal{V}^1_{1+\delta}(\Pi)$ where $\delta \in (0, 1/2)$ is such that the interval $\upsilon = \big(\frac12(1-d) - 1 - \delta, \frac12(1-d) - \delta\big)$ contains the exponent $\Lambda_0^-(3) = -2$ for $d = 3$ and no exponent (2.19) for $d > 3$. Since $r^{\Lambda_0^-} \notin H^1(\Pi)$, see (2.20), Theorem 3.5 shows that $u \in \mathcal{V}^1_\delta(\Pi)$. Applying this theorem several times we diminish the weight index and arrive at the inclusion $u \in \mathcal{V}^1_{1-\delta_0+(1-d)/2}(\Pi)$ with a small $\delta_0 > 0$, e.g. $\delta_0 = 1/4$. Then the interval (3.5) with $\beta = 1-\delta_0+(1-d)/2$, $\gamma = \beta - 1$, includes the exponent $\Lambda_0^+ = 0$ but no exponent $\Lambda_k^+(d)$, $k \geq 1$, see Section 2.2. Finally, the representation (3.17) is valid with $\mathbf{U} = C_0 = u(\mathcal{O})$. We set

$$\widetilde{u}^0 = u - \chi C_0 \Big(1 + \sum_{p=1}^{N} U^{p0} + U_\perp^{N+1,0} \Big) \ , \tag{3.27}$$

and get the inclusion $\widetilde{u}^0 \in \mathcal{V}^1_{-\delta_0+(1-d)/2}(\Pi)$. If $N = 0$, the goal is achieved. Otherwise, we observe that the right hand sides $\widetilde{f}^0$ and $\widetilde{g}^0_\pm$ of the problem (3.7) for the function (3.27) belong to $\mathcal{L}^2_{\sigma+1}(\Pi)$ and $\mathcal{L}^2_\sigma(\varpi_\pm)$, respectively. Then, we again use Theorem 3.5 with $\beta = \beta_0$, $\beta = \beta_1$, and

$$\beta_k = -k - \delta_k + (1-d)/2 \ , \ 0 < \delta_k < \min\{\Lambda_k(d) - k - 1, \Lambda_{k+1}(d) - k\}$$

(the last minimum is positive due to the calculations (2.21)–(2.22)). Consequently, $\widetilde{u}^1 \in \mathcal{V}^1_{\beta_1}(\Pi)$, where

$$\widetilde{u}^m = u - \chi \sum_{k=0}^{m} C_k \Big(U^{0k} + \sum_{p=1}^{N-[k/2]} U^{pk} + U_\perp^{N+1-[k/2]k} \Big), \tag{3.28}$$

$[k/2]$ denotes the integer part of $k/2$, and $U^{pk} = U_0^{pk} + U_\perp^{pk}$ are terms in the asymptotic tails (2.23) initiated by the power-law solution $U^{0k}(y) = C_k r^{\Lambda_k^+(d)} \Phi_k^+(\theta)$ and constructed in Section 2.3.

Repeating the above consideration several times we conclude that the function (3.28) with $m = N$ belongs to $\mathcal{V}^1_\sigma(\Pi)$ with $\sigma = \beta_N$ and $\delta = \delta_N$ in (3.26).

Theorem 3.7. *Let $d \geq 3$. For any $n, N \in \mathbb{N}$, the eigenfunction u_n of the problem* (1.3), (1.4) *has the asymptotic form*

$$u_n(x) = \chi(x) \sum_{k=0}^{2N} c_k \Big(r^{\Lambda_k^+(d)} \Phi_k^+(\theta) + \sum_{p=1}^{N-[k/2]} U^{pk}(y,\zeta) \Big) + \widetilde{u}_n^N(x) \ , \tag{3.29}$$

where χ is the cut-off function (3.16)*, c_k are some constants, the remainder $\widetilde{u}_n^N$ satisfies* (3.26)*, and other expressions have been determined in Sections 2.2 and 2.3.*

We summarize that by Section 2.2, $\Lambda_0^+(d) = 0$ and $0 < \Lambda_1^+(d) < 1 < \Lambda_k^+(d)$ for all $d \geq 3$, $k \geq 2$, so that the gradient of $u_n(x)$ has the singularity of order $r^{\Lambda_1^+(d)-1}$ claimed in the introduction, and in particular we have $\Lambda_1^+(3) - 1 = \sqrt{2} - 2 \approx -0.586$ in dimension $d = 3$.

In comparison with (3.28), some terms were not included in (3.29), but we will not need to pay attention to them in the final assertion Theorem 3.9.

3.4. **Weighted pointwise estimates for the remainders.** Since $u_n \in C^\infty(\overline{\Omega} \backslash \mathcal{O})$, we only need to derive pointwise estimates in the gap Π. Moreover, all terms in the series (2.23) are infinitely differentiable in the variables $y \in \mathbb{B}_\bullet = \mathbb{B} \setminus \mathcal{O}$ and $z \in [-H_-(y), H_+(y)]$, hence, it is sufficient to deal with the remainder $\widetilde{u}^N \in \mathcal{V}^1_\sigma(\Pi)$, see (3.28). This is a solution of the problem (3.7) with the right hand sides $\widetilde{f}^N$ and $\widetilde{g}^N_\pm$, which meet the estimates

$$\begin{aligned} &|\nabla_x^k \widetilde{f}^N(x)| \leq c_{Nk} r^{2N-3-2k} \ , \ x \in \overline{\Pi} \setminus \mathcal{O} \ , \\ &|\nabla_y^k \widetilde{g}^N_\pm(y)| \leq c_{Nk} r^{2N-1-2k} \ , \ x \in \varpi_\pm \ , \ k \in \mathbb{N}_0, \end{aligned} \tag{3.30}$$

where $\nabla_x^k v$ denotes the collection of all partial derivatives of order k of a function v and c_{Nk} are some positive constants.

Remark 3.8. The estimates (3.30) are quite rough, although they are sufficient for our purposes, since in this section we do not distinguish between the differentiation in the longitudinal $y-$ and transversal $z-$variables in the gap, cf. (2.10), and since we have excluded the last terms of (3.28) in the sum (3.29). The crucial point is that N is an arbitrary integer in Theorems 3.6 and 3.7, so that taking it large, applying "inaccurate" Hölder estimates and finally moving the "extra" terms to the remainder, we obtain "precise" pointwise estimates. In this way, we do not need to introduce weighted Hölder norms, instead write local estimates (3.33), which lose information on the differentiability properties of the solution. Also the arbitrariness of $k \in \mathbb{N}$ helps in this respect. $\boxtimes$

Let $y^0 \in \mathbb{B}$ be a point such that $r_0 = |y^0|$ is small, in particular, $4r_0 < R$. We introduce two cells $\theta_1 \subset \theta_2$,

$$\theta_p = \{x \in \Pi \, : \, |y_n - y_n^0| < p^2 r_0^2, \ n = 1, \dots, d-1\} \ , \quad p = 1, 2. \tag{3.31}$$

Owing to (1.7) and (1.11), the coordinate change $x \mapsto X = (Y, Z) = r_0^{-2}(y - y^0, z)$ transforms the cells (3.31) into the sets Θ_p having volume of order 1 as $r_0 \to 0$ and cubical cross-section $\square_p = \{Y \, : \, |Y_n| \le p^2, n = 1, \dots, d-1\}$ and "almost flat" bases

$$\Sigma_p^\pm = \Big\{X \, : \, Y \in \square_p, Z = \pm r_0^{-2} H_\pm(y^0 + r_0^2 Y) = \pm \frac{1}{2R_\pm} + O(r_0^2)\Big\},$$
$$\big|\nabla_Y^k \big(r_0^{-2} H_\pm(y^0 + r_0^2 Y)\big)\big| \le c_k r_0^2 \ , \quad Y \in \square_p.$$

These imply that one can choose a constant c_l independently of y^0 in the weakened local estimate [1]

$$\sum_{j=0}^{l+2} \sup_{\Theta_1} \big|\nabla_X^j U(X)\big| \le c_l \ \Big(\ r_0^4 \sum_{j=0}^{l+1} \sup_{\Theta_2} \big|\nabla_X^j \mathcal{F}(X)\big|$$
$$+ \ r_0^2 \sum_{\pm} \sum_{j=0}^{l+2} \sup_{\Sigma_2^\pm} \big|\nabla_Y^j \mathcal{G}_\pm(X)\big| + \|U; L^2(\Theta_2)\|\Big) \ \text{ with } l \in \mathbb{N}_0. \tag{3.32}$$

Here, the function $U(X) = \widetilde{u}(y^0 + r_0^2 Y, r_0^2 Z)$ satisfies the problem

$$-\Delta_x U - r_0^4 \lambda U = r_0^4 \mathcal{F} \ \text{ in } \Theta_2 \ , \quad \partial_{\nu^\pm(X)} U = r_0^2 \mathcal{G}_\pm \ \text{ on } \Sigma_2^\pm,$$

where $\mathcal{F}(X)$ and $G_\pm(Y)$ are the functions $\widetilde{f}$ and $\widetilde{g}_\pm$ written in the stretched coordinates. Returning to the original coordinates, (3.32) yields the inequality

$$\sum_{j=0}^{l+2} r_0^j \sup_{\theta_1} |\nabla_x^j \widetilde{u}(x)| \le c_l \Big(\sum_{j=0}^{l+1} r_0^{4+j} \sup_{\theta_2} |\nabla_x^j \widetilde{f}(x)|$$
$$+ \sum_{\pm} \sum_{j=0}^{l+2} r_0^{2+j} \sup_{\varsigma_2^\pm} |\nabla_x^j \widetilde{g}_\pm(y)| + r_0^{-d} \|\widetilde{u}; L^2(\theta_2)\|\Big). \tag{3.33}$$

We emphasize that the local estimate (3.32) was weakened by not using the Hölder seminorm for $\mathcal{U}$ on the right and by estimating the Hölder seminorms of $\mathcal{F}$ and $\mathcal{G}_\pm$ by higher-order derivatives on the right.

We remark that for some constants $C > c > 0$ we have

$$cr \le r_0 < Cr \quad \text{for } x \in \theta_2$$

so that we can replace r_0 by $r = |y|$ and thus obtain after multiplying by $r_0^{\sigma-1+d}$ the rough estimate

$$\sum_{j=0}^{l+2} \sup_{\theta_1} |r^{2(\sigma-1+d+j)} \nabla_x^j \widetilde{u}(x)| \le c\Big(\sum_{j=0}^{l+1} \sup_{\theta_2} |r^{2(\sigma+3+d-j)} \nabla_x^j \widetilde{f}(x)|$$

$$+ \sum_{\pm} \sum_{j=0}^{l+2} \sup_{\zeta_2^{\pm}} |r^{2(\sigma+d-j)} \nabla_y^j \widetilde{g}_{\pm}(y)| + \|\widetilde{u}; \mathcal{L}^2_{\sigma-1}(\theta_2)\| \Big).$$

The last weighted Lebesgue norm is bounded due to the results above, and the boundedness of the weighted maxima of $\nabla_x^j \widetilde{f}$ and $\nabla_y^j \widetilde{g}_{\pm}$ can always be achieved by taking into account sufficiently many terms in the asymptotic representations constructed in Section 2.3. Recalling Remark 3.8 we formulate the final assertion of our paper.

Theorem 3.9. *The remainder $\widetilde{u}_n^N$ in the expansion* (3.24) *($d = 2$) and* (3.29) *($d \ge 3$) for the eigenfunction u_n of the problem* (1.3), (1.4), *satisfies the estimate*

$$\big|\nabla_y^p \partial_z^q \widetilde{u}_n^N(x)\big| \le c_{pqn} |x|^{N+\delta_n-p-2q} \ , \quad x \in \Omega \ , p, q \in \mathbb{N}_0 \ ,$$

where $\delta_n > 0$ and $c_{pqm} > 0$ are some constants.

In view of Section 2.5, this theorem indeed proves that in dimension $d = 2$ all eigenfuntions and their derivatives are bounded in the domain Ω.

In dimension $d \ge 3$ the gradient $\nabla_x u(x)$ includes the components (see Section 2.2)

$$\partial_r u_n(x) = \Lambda_1^+(d) r^{\Lambda_1^+(d)-2} \sum_{j=1}^{d-1} c_j y_j + O(r^{\Lambda_2^+(d)-1}),$$

$$r^{-1} \widetilde{\nabla}_\theta u_n(x) = r^{\Lambda_1^+(d)-1} \sum_{j=1}^{d-1} c_j \widetilde{\nabla}_\theta (r^{-1} y_j) + O(r^{\Lambda_2^+(d)-1}),$$

$$\partial_z u_n(x) = r^{\Lambda_1^+(d)+2} H(y)^{-1} \sum_{j=1}^{d-1} c_j \partial_\zeta \Phi_\perp^{1j}(\theta, \zeta) + O(r^{\Lambda_2^+(d)}),$$

where $r^{\Lambda_1^+(d)+2} \Phi_\perp^{1j}(\theta, \zeta)$ is the function (2.17) constructed from the power-law solutions $r^{\Lambda_1(d)} \Phi^{0j}(\theta)$ with $\Phi^{0j}(\theta) = r^{-1} y_j$, the trace of the linear solution y_j on the sphere $\mathbb{S}^{d-1}$. Thus, the vector function $\nabla_x u_n$ is bounded in Ω, if and only if the coefficients $c_1, \dots, c_{d-1}$ vanish. This may happen for eigenfunctions which are rotationally symmetric with respect to the z-axis of the domain (2.14).

4. Other shapes.

4.1. **Nested kissing balls.** In the geometric situation of Fig. 4.1 we have $R_- < 0$ and $0 < R_+ < |R_-|$, and only minor changes are needed to treat this case. For example, we have

$$\pm H_{\pm}(y) = R_{\pm}^2 - \sqrt{R_{\pm}^2 - |y|^2} \ , \quad A^0 = \frac{1}{2}\big(R_+^{-1} - |R_-|^{-1}\big) \ , \ a^0 = \frac{1}{4}\big(R_+^{-1} + |R_-|^{-1}\big)$$

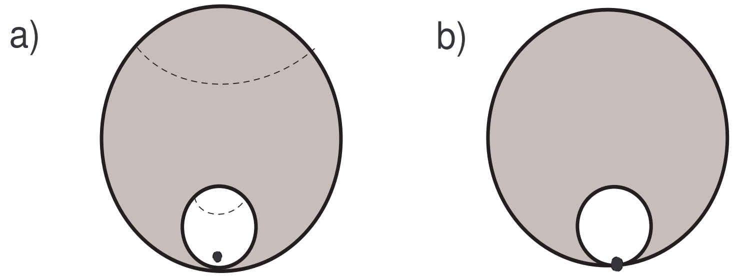

FIGURE 4.1. Nested kissing balls a) in $d = 3$, b) in $d = 2$.

instead of the old formulas (1.11) and (2.12). All conclusions remain valid, literally.

4.2. **Ellipsoids.** Let us consider the case that the balls (1.1) are replaced by the ellipsoidal cavities

$$E^{\pm} = \Big\{ x = (y, z) \ : \ \frac{1}{\ell_{d\pm}^2}(z - \ell_{d\pm})^2 + \sum_{n=1}^{d-1} \frac{y_n^2}{\ell_{n\pm}^2} < 1 \Big\}, \ \ell_{p\pm} > 0,$$

in the definition (1.2) of the domain Ω, where the spectral Neumann problem is posed. Then, the thickness function $H(y) = H_+(y) + H_-(y)$ has the terms

$$H_{\pm}(y) = \ell_{d\pm}\Big(1 - \sqrt{1 - \ell_{d\pm}^{-2}\big(\ell_{1\pm}^{-2} y_1^2 + \ldots + \ell_{d-1\pm}^{-2} y_{d-1}^2}\Big),$$

and we note that neither $H(y)$ nor the coefficient $H^0(y) = r^2 A^0(\theta)$ in the limit differential operator in $\mathbb{R}_\bullet^{d-1}$ depends on the angular variable $\theta \in \mathbb{S}^{d-2}$, if and only if

$$\ell_{1\pm} = \ldots = \ell_{d-1\pm}, \tag{4.1}$$

i.e., the ellipsoids are rotationally symmetric with respect to the z-axis. In any case the general scheme and main results remain the same as for the case of balls; however, in the case $d = 3$, in order to keep the simplicity of the calculations of Section 2.2, one certainly has to pose the algebraic restriction (4.1), otherwise it is necessary to solve a second order differential equation with variable coefficients, which cannot be done explicitly. Also, in dimension $d = 2$ the formula (4.1) loses its meaning and the gap Π splits into two cuspidal peaks, Fig. 1.1.b), and $\theta \in \mathbb{S}^0 = \{\pm 1\}$ so that the material in Sections 2.5 and 3.3 remains unchanged.

4.3. **Ball touching a paraboloid.** Let us define the gap Π in a domain Ω by

$$y \in \mathbb{B}_\bullet \ , \quad P|y|^2 < z < R - \sqrt{R^2 - |y|^2}$$

where $R > 0$ and $P \le (2R)^{-1}$. In the case $P < (2R)^{-1}$ we have

$$H(y) = \big((2R)^{-1} - P\big)|y|^2 + O(|y|^4) \ , \quad |y| \to 0 \ , .$$

and all calculations and conclusions of Sections 2 and 3 remain valid as such. However, if $P = (2R)^{-1}$, the thickness function

$$H(y) = (4R^3)^{-1}|y|^4 + O(|y|^6)$$

decays faster as $y \to \mathcal{O}$. Our general scheme of asymptotic analysis for the eigenfunctions of the problem (1.3), (1.4) still works, but the detailed calculations should be revised. In particular, the exponents of the power-law solutions of the limit equation

$$-\nabla_y \cdot \big(|y|^4 \nabla_y U^0(y)\big) = 0 \ , \quad y \in \mathbb{R}^{d-1}_\bullet,$$

are given by

$$\Lambda_k^\pm(d) = \frac{1}{2}\Big(-1 - d \pm \sqrt{(1+d)^2 + 4k(k+d-3)}\Big) \ , \quad k \in \mathbb{N}_0,$$

instead of (2.19). It is remarkable that in dimension $d = 3$ the singularity with exponent $\Lambda_1^+(3) - 1 = \sqrt{5} - 3 \approx -0.764$ is much stronger than $\sqrt{2} - 2 \approx -0.586$ of the case of kissing balls, see (2.21).

4.4. **Ellipsoid touching a paraboloid.** Let $d = 3$ and

$$\Pi = \Big\{ x = (y,z) \, : \, |y| < R, \ P|y|^2 < z < \ell_3\Big(1 - \sqrt{\ell_1^2 y_1^2 + \ell_2^2 y_2^2}\Big), \ell_j > 0 \Big\},$$

where $P = (2\ell_1^2)^{-1}\ell_3$ but $P < (2\ell_2^2)^{-1}\ell_3$. Then, the dependence of the thickness function

$$H(y) = \big((2\ell_2^2)^{-1}\ell_3 - P\big)y_2^2 + O(|y|^4)$$

on y_2 and y_1 is of different homogeneity orders. Thus, our techniques fail in this case. The question on the asymptotics of the Neumann eigenfunctions at the points of tangency of a symmetric paraboloid and an asymmetric ellipsoid remains open.

4.5. **Two tori.** Let $d = 3$ and let $T^\pm$ be the torus with the guide circle $\{x \, : \, z = \pm R_0, \ y_1^2 y_2^2 = R^2\}$ and the generating circle $\{x \, : \, y_1 = 0, |z - R_0|^2 + y_2^2 = R_0^2\}$, where $R > R_0$. Then, the gap

$$\Pi = \Big\{ x \, : \, |z - R| < R_0, \ |z| < R_0 - \sqrt{R_0^2 - |r - R|^2} \Big\}$$

of the domain $\Omega = \Omega_0 \setminus \big(\overline{T^+} \cup \overline{T^-}\big)$, Fig. 1.3.b), has a degeneration line $\{x \, : \, z = 0, |y| = r\}$, which divides Π into two subdomains with cuspidal edges. Such irregular submanifolds for the Neumann problem have not been investigated yet, although our calculations in Section 2.5 allow us to state the hypothesis that the eigenfunctions of the problem (1.3), (1.4) together with their derivatives are bounded.

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